Equivalent Conditions for Synchronization of Identical Linear Systems and Application to Quality-Fair Video Delivery

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***Abstract*—This paper addresses the problem of consensus of multi-agent systems, consisting of a set of identical continuous- or discrete-time systems, connected through a network with fixed topology. The information exchanged over the communication network is the output of each system. First, necessary and sufficient conditions are given for the uniform global exponential stability of the consensus set. Then, the quality-fair delivery of media contents is dealt with as an application of the presented equivalent conditions, which lead to a control design paradigm consisting in a suitable linear static output feedback design problem. This problem is solved proposing an iterative design technique based on Linear Matrix Inequalities (LMIs), and its effectiveness is demonstrated via simulation results, where we compare our results with pre-existing approaches.**

I. Introduction

In recent years, consensus and synchronization problems among multi-agent systems have received an intensive interest in the literature, due to the variety of applications in many different areas including cooperative control of unmanned areal vehicles, formation control of mobile robots and communication among sensor networks. See, e.g., [11], [13], [16], [17], [24], [29]. Specifically, *consensus* refers to agents coming to a global agreement on a state value, by the exchange of information modeled by some communication graph. It has been shown in [22], [23] that mild assumptions on graph connectivity ensure to uniformly exponentially reach consensus, see also [16], [25]. Compared to consensus problems, the *synchronization* literature refers to agents moving toward a common trajectory in the configuration space [12], [35], [37], [38].

Consensus algorithms are primarily studied when the agents' open-loop internal dynamics are described by an integrator chain (e.g. single- or double-integrator models [25], [28]). Recently, for full generality, the consensus problem has been investigated considering agents modeled by general linear time-invariant (LTI) systems [34], [45], [51]. Consensus and synchronization problems are extensively studied in the literature for identical multi-agent systems, see, *e.g.*, [10], [11], [27], [43]. In [44] a state-feedback consensus protocol is proposed for linear multi-agent systems with switching topology. In [50] sufficient and necessary conditions on consensus of linear multi-agent systems are provided by using the full state information of the agents. In [42], a linear quadratic regulator (LQR) based optimal control approach was used for the controller design via state-feedback information. When the full state is not available, an observer can be used to estimate the states [15], [20], [32], which makes the control architecture more complex. To overcome this problem, output-feedback based control may provide satisfactory solutions and some methods have been given [19], [21], [52]. A low gain approach to dynamic output feedback compensator design for consensus was given in [34]. Reference [19] proposed an observer-type consensus protocol designed using Finsler's lemma and LMI techniques. Consensus analysis and design via LMI numerical procedures are presented in [36], [40], [48] for interconnected systems under communication delays.

Despite the above mentioned extensive amount of work in the field, one cannot find a general theorem about synchronization of identical linear systems without any structure. Since for synchronization solutions may diverge while synchronizing, one should pay special attention to the fact that the attractor (the synchronization set where all the states concide) is unbounded, and then uniformity of stability and attractivity is nontrivially shown. Establishing equivalent conditions for this property, also involving strict Lyapunov functions, is the first contribution of this paper.

A second contribution of this paper is the use of the necessary and sufficient synchronization conditions mentioned above to the design of a quality-fair video streaming system to several users sharing some common wireless ressource, considering the model proposed in [3], [4]. When several users share some communication link to get streamed video contents, simple bit-rate or bandwidth fair allocation strategies are usually inappropriate. Such strategies are agnostic of the rate-quality characteristics of the delivered contents. Rather static video contents such as news may be efficiently delivered with a moderate bit rate, that would be insufficient to enjoy an action motion picture of decent quality. This has motivated the recent development of quality-fair video delivery techniques, such as [2], [6], [7], [18]. For example, [6] considers an utility max-min fair resource allocation, which tries to maximize the worst utility. Nevertheless, it does not consider the temporal variability of the rate-utility characteristics of the contents, or the delays introduced by the network and the buffers of the delivery system. In [18], a content-aware distortion-fair video delivery scheme is proposed, assuming that the characteristics of the video frames are known in advance, which restricts its usage to the streaming of stored videos. In [7], a Lagrangian optimization framework is considered to maximize the sum of the achievable rates while minimizing the distortion difference

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among streams. This requires to gather all rate-utility characteristics of the streams at the control unit. The user experience is accurately modeled in [5] using the empirical cumulative distribution function of the predicted video quality. Feedback control techniques have been considered in [3], [4] to reach a quality fairness among users, while controlling the level of the buffers in the network or the buffering delay. However, the tuning of the parameters is nontrivial and has been performed heuristically in these works.

To the best of our knowledge, it is the first time that such parallel video delivery problem with ressource and quality-faireness constraints is studied with consensus techniques. Our proposed approach leads to an attractive linear static output feedback design problem that ensures suitable performance guarantees in addition to consensus. Preliminary results in this direction have been reported in [8]. We include here a more detailed description of our approach, proofs that were missing in [8], a systematic synthesis procedure for the linear static output feedback design problems, and revised simulation tests.

Summarizing, the contribution of this paper, is twofold. First, we derive a general Lyapunov-based synchronization result for continuous- and discrete-time identical linear systems. More precisely, the equivalence is proven among several well known conditions and uniform global exponential stability of the consensus set. Second, these results are shown to lead to a systematic design technique based on iterative LMIs for an optimized selection of the PI gains in the experimental application modeled in [3], [4] maximizing the convergence rate to the synchronization set while guaranteeing it uniform global exponential stability. A further illustration of our results is given in our simulation section, where we compare our results to the ones obtained with the PI tuning of [3], [4] which followed a heuristic technique. Our results on necessary (and sufficient) stability conditions predict that the heuristic selection of [4] is non stabilizing, and indeed simulation results show evidence of instability that disappears with our selection.

The paper is organized as follows. Section II presents the linear consensus problem and gives the main result. In Section III the quality-fair video delivery problem is cast as a distributed consensus problem. In Section IV a systematic method to design the controller gains is proposed using Finsler's lemma and LMI techniques. The effectiveness of this method is illustrated on experimental tests in Section V. Concluding remarks end the paper.

Notation. We use $x^+ = x^+(j) = x(j+1)$ to denote the push-forward operator, $\forall j \in \mathbb{Z}_+$, $x^d = x^d(j) = x(j-1)$ to denote the one step delay operator, and $x^{dd} = x^{dd}(j) = x(j-2)$ to denote the two steps delay operator. We denote with $\mathbf{1}_N$ the N dimensional (column) vector, for which entries are all 1. For any square matrices $A_1, \ldots, A_N$, the notation $\operatorname{diag}(A_1, \ldots, A_N)$ indicates the block diagonal matrix whose diagonal blocks are $A_1, \ldots, A_N$. We denote with $\mathbb{C}_{\leq\beta}$ the set of the complex numbers with modulus less or equal to β. The symbol $\mathbf{0}_{m,n}$ denotes the zero matrix of size $m \times n$

II. NECESSARY AND SUFFICIENT CONDITIONS FOR SYNCHRONIZATION OF IDENTICAL LINEAR SYSTEMS

Consider N identical dynamical systems, governed by:

$$\begin{array}{rcl} \delta x_i &=& A x_i + B u_i \\ y_i &=& C x_i \end{array} \qquad i = 1, \ldots, N \tag{1}$$

where $\delta x = \dot{x}$ for continuous-time and $\delta x = x^+$ for discrete-time. In (1), $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}$, $y_i \in \mathbb{R}$. Consider the interconnection:

$$u = -Ly, \tag{2}$$

where $u = [u_1 \ldots u_N]^\top \in \mathbb{R}^N$, $y = [y_1 \ldots y_N]^\top \in \mathbb{R}^N$ and $L = L^\top \in \mathbb{R}^{N \times N}$ is the graph Laplacian of the network. Also denote the eigenvalues of L as $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1}$, where it is emphasized (see [11]) that L has always an eigenvalue at zero, that corresponds to the eigenvector $\mathbf{1}_N$.

With the goal of establishing synchronization among systems (1), let us introduce the following consensus set:

$$\mathcal{A} := \{x : x_i - x_j = 0, \forall i, j \in \{1, \ldots, N\}\}. \tag{3}$$

Let also recall that given a set $\mathcal{X}$, $|x|_\mathcal{X} = \inf_{y \in \mathcal{X}} |x - y|$. Our main consensus result, given below, relies on suitable properties of quadratic functions with respect to set $\mathcal{A}$ in (3). These are given in the next lemma, whose proof is reported in the appendix.

Lemma 1: Consider any unitary matrix $T \in \mathbb{R}^{N \times N}$ whose first column is given by $\frac{1}{\sqrt{N}}\mathbf{1}_N$ and the diagonal matrix $\Delta = I_N - e_1 e_1^\top$, where $e_1 = [1\ 0\ \ldots\ 0]^\top \in \mathbb{R}^N$ is the first element of the Euclidean basis. Then there exist scalars $c_1, \bar{c}_1, c_2, \bar{c}_2 > 0$ such that for any $n \in \mathbb{N}$ and any $x \in \mathbb{R}^{Nn}$, where $x_k \in \mathbb{R}^n$, $\forall k = 1, \ldots, N$, we have:

$$\bar{c}_1 |x|_\mathcal{A}^2 = c_1 \sum_{k=2}^{N} |x_1 - x_k|^2 \leq x^\top (T \Delta T^\top \otimes I_n) x, \tag{4a}$$

$$\bar{c}_2 |x|_\mathcal{A}^2 = c_2 \sum_{k=2}^{N} |x_1 - x_k|^2 \geq x^\top (T \Delta T^\top \otimes I_n) x. \tag{4b}$$

Based on Lemma 1, we can now state a set of necessary and sufficient conditions for synchronization of identical linear systems. The proof combines the stability results in [11] with the established output feedback coupling approach of [33]. Parts of the following result can be found in the literature: for example, implication (i) $\implies$ (iii) is established in an equivalent formulation in [11, Theorem 3], [49, Theorem 1] and [34, Theorem 1] for the convergence part. In [46], the equivalence (i) $\iff$ (iv) is proven for the continuous-time case.

Theorem 1: Consider the continuous-time [respectively, discrete-time] system (1), (2) and the attractor $\mathcal{A}$ in (3). The following statements are equivalent:

(i) Matrices

$$A_k := A - \lambda_k BC, \ k = 1, \ldots, N-1 \tag{5}$$

are Hurwitz [respectively, Schur-Cohn].

(ii) There exists a strict quadratic Lyapunov function $V(x)$ satisfying:

$$\bar{c}_1 |x|_\mathcal{A}^2 \leq V(x) \leq \bar{c}_2 |x|_\mathcal{A}^2, \tag{6a}$$

$$\dot{V}(x) \leq -\bar{c}_3 \, |x|^2_{\mathcal{A}} \,, \tag{6b}$$

$$[\text{respectively, } \Delta V(x) \leq -\bar{c}_3 \, |x|^2_{\mathcal{A}} \,], \tag{6c}$$

for suitable positive constants $\bar{c}_1$, $\bar{c}_2$ and $\bar{c}_3$, where $|x|_{\mathcal{A}}$ denotes the distance of x from the set $\mathcal{A}$.

(iii) The closed attractor $\mathcal{A}$ in (3) is uniformly globally exponentially stable for the closed loop (1), (2).

(iv) The closed loop (1), (2) is such that the sub-states x_i uniformly globally exponentially synchronize to the unique solution to the following initial value problem:

$$\delta x_\circ = A x_\circ, \quad x_\circ(0) = \frac{1}{N}\sum_{k=1}^{N} x_k(0). \tag{7}$$

Proof: we first show a preliminary transformation, then we prove the theorem in four steps: (i) $\implies$ (ii), (ii) $\implies$ (iii), (iii) $\implies$ (iv), and (iv) $\implies$ (i).

Preliminary transformation. Let us define the extended state vector $x = \left[x_1^\top \ldots x_N^\top\right]^\top$ and rewrite interconnection (1), (2) in the following compact form:

$$\begin{aligned} \delta x &= (I_N \otimes A)x + (I_N \otimes B)u && \text{(8a)}\\ y &= (I_N \otimes C)x && \text{(8b)}\\ u &= -(L \otimes C)x = -(I_N \otimes C)(L \otimes I_n)x, && \text{(8c)} \end{aligned}$$

where $I_N \otimes A \in \mathbb{R}^{Nn \times Nn}$, $I_N \otimes B \in \mathbb{R}^{Nn \times N}$, $I_N \otimes C \in \mathbb{R}^{N \times Nn}$ and $L \otimes I_n \in \mathbb{R}^{Nn \times Nn}$. Since matrix L is symmetric, there exists a unitary matrix $T \in \mathbb{R}^{N \times N}$ (namely a matrix satisfying $T^\top T = I_N$) that diagonalizes L. In particular, let us pick T such that:

$$\Lambda = T^\top L T = \operatorname{diag}(0, \lambda_1, \ldots, \lambda_{N-1}) \,. \tag{9}$$

Since the upper-left entry of Λ is zero, we may select T such that its first column corresponds to the eigenvector $t_0 = \frac{1}{\sqrt{N}}\mathbf{1}_N$ associated to the zero eigenvalue $\lambda_0 = 0$ of L. Furthermore, it is easily checked that $T \otimes I_n$ transforms $L \otimes I_n$ into $\Lambda \otimes I_n$. Indeed, using the associative property of the Kronecker product we get $(T \otimes I_n)^\top (L \otimes I_n)(T \otimes I_n) = (T^\top L T \otimes I_n) = \Lambda \otimes I_n$. Let us now introduce the similarity transformation $\bar{x} = (T^\top \otimes I_n)x$. Then dynamics (8a) reads:

$$\begin{aligned} \delta x &= (T \otimes I_n)^{-1}(I_N \otimes A)(T \otimes I_n)\bar{x} + && \text{(10a)}\\ &\quad +(T \otimes I_n)^{-1}(I_N \otimes B)u \\ y &= (I_N \otimes C)(T \otimes I_n)\bar{x} && \text{(10b)}\\ u &= -(I_N \otimes C)(L \otimes I_n)(T \otimes I_n)\bar{x}. && \text{(10c)} \end{aligned}$$

Substituting in (10) the control law (10c) into (10a) and using the associative property of the Kronecker product we obtain:

$$\delta \bar{x} = \bar{A}\bar{x}, \tag{11}$$

where the state matrix A can be computed as:

$$\begin{aligned} \bar{A} &= (T^{-1} T \otimes A) \\ &\quad -(T \otimes I_n)^{-1}(I_N \otimes B)(I_N \otimes C)(L \otimes I_n)(T \otimes I_n) \\ &= (I_N \otimes A) - (T^{-1} L T \otimes BC) \\ &= (I_N \otimes A) - (\Lambda \otimes BC) \\ &= (I_N \otimes A) - (I_N \otimes BC)(\Lambda \otimes I_n), \end{aligned} \tag{12}$$

which has the following block diagonal structure:

$$\bar{A} = \operatorname{diag}(A, A_1, \ldots, A_{N-1}) \,, \tag{13}$$

by using the definitions in (5). *Proof of (i) $\implies$ (ii).* By assumption (5), we have that there exist matrices P_k, $k = 1, \ldots, N-1$, such that:

$$A_k^\top P_k + P_k A_k = -I_n, \; k = 1, \ldots, N-1 \,, t \in \mathbb{R}, \text{ or} \tag{14a}$$

$$A_k^\top P_k A_k - P_k = -I_n, \; k = 1, \ldots, N-1 \,, t \in \mathbb{Z}. \tag{14b}$$

Consider the block diagonal matrix $\bar{P} = \operatorname{diag}(0, P_1, \ldots, P_{N-1})$ and define the Lyapunov function candidate:

$$V(x) = \underbrace{x^\top (T \otimes I_n)}_{\bar{x}^\top} \bar{P} \underbrace{(T^\top \otimes I_n)}_{\bar{x}} = \sum_{k=1}^{N-1} \bar{x}_k^\top P_k \bar{x}_k. \tag{15}$$

Then, from equations (11) and (14a) or (11) and (14b) it follows that:

$$\begin{aligned} \dot{V}(x) &= \sum_{k=1}^{N-1} \bar{x}_k^\top (P_k A_k + A_k^\top P_k)\bar{x}_k = -\sum_{k=1}^{N-1} \bar{x}_k^\top \bar{x}_k, \text{ or} \\ \Delta V(x) &= \sum_{k=1}^{N-1} \bar{x}_k^\top (A_k^\top P_k A_k - P_k)\bar{x}_k = -\sum_{k=1}^{N-1} \bar{x}_k^\top \bar{x}_k. \end{aligned} \tag{16}$$

To prove (6), we use Lemma 3 (which is postponed in the appendix), after noticing that matrix T introduced in the preliminary step of the proof satisfies the assumption of the lemma. Then we also observe that, using matrix $\Delta = \operatorname{diag}\{0, 1, \ldots, 1\}$ defined in Lemma 1, we have:

$$\begin{aligned} \sum_{k=1}^{N-1} \bar{x}_k^2 &= \bar{x}^\top (\Delta \otimes I_n)\bar{x} \\ &= \left((T \otimes I_n)x\right)^\top (\Delta \otimes I_n)(T \otimes I_n)x \\ &= x^\top (T^\top \Delta T \otimes I_n)x. \end{aligned} \tag{17}$$

Therefore, using (17), positive definiteness of P_k, $k = 1, \ldots, N-1$, definition (15) and Lemma 1, we obtain:

$$\begin{aligned} V(x) &\leq \underbrace{\max_{h \in \{1,\ldots,N-1\}} \lambda_{\max}(P_h)}_{\bar{p}} \sum_{k=1}^{N-1} |x_k|^2 \\ &= \bar{p} x^\top (T^\top \Delta T \otimes I_n)x \leq c_2 \bar{p} \, |x|^2_{\mathcal{A}} \,, \\ V(x) &\geq \underbrace{\min_{h \in \{1,\ldots,N-1\}} \lambda_{\min}(P_h)}_{\underline{p}} \sum_{k=1}^{N-1} |x_k|^2 \\ &= \underline{p} x^\top (T^\top \Delta T \otimes I_n)x \geq c_1 \underline{p} \, |x|^2_{\mathcal{A}} \,. \end{aligned} \tag{18}$$

Thus (6a) is proven with $\bar{c}_1 = c_1 \underline{p}$ and $\bar{c}_2 = c_2 \bar{p}$. Finally, using (16), (17) and Lemma 1 we get:

$$\dot{V}(x) \leq -x^\top (T^\top \Delta T \otimes I_n)x \leq -c_1 \, |x|_{\mathcal{A}} \,, \tag{19}$$

which coincides with (6b) with $\bar{c}_3 = c_1$. Note that, the same majoration holds for the discrete-time case, so that (6c) is satisfied.

Proof of (ii) $\implies$ (iii). Based on (6), the uniform global exponential stability of $\mathcal{A}$ in (3) follows from standard Lyapunov results (see, *e.g.*, the discrete- and continuous-time special cases of the hybrid results in [41, Theorem1]).

Proof of (iii) $\implies$ (iv). Consider the dynamics of the state $x_\circ(t) := \frac{1}{N}\sum_{k=1}^{N} x_k(t)$ and note that from (1):

$$\delta x_\circ(t) = \frac{1}{N}\sum_{k=1}^{N}\delta x_k(t) = A\sum_{k=1}^{N} x_k(t) + B\sum_{k=1}^{N} u_k(t)$$
$$= Ax_\circ(t) + B\underbrace{\mathbf{1}_N^\top L}_{=0} y = Ax_\circ(t), \tag{20}$$

where $\mathbf{1}_N^\top L = 0$ due to well known properties of Laplacian matrices. Then $x_\circ$ evolves autonomously according to (7) and corresponds to the average of states x_k. Since from (ii) $\implies$ (iii) we know that states x_k exponentially synchronize to some consensus, then they must synchronize to their average value that is $x_\circ$.

Proof of (iv) $\implies$ (i). We prove this step by contradiction. Assume that one of matrices A_k in (5) is not Schur-Cohn, and assume without loss of generality that it is A_{N-1}. Consider the coordinate system in (11) with (13). Then, from the block diagonal structure of $\bar{A}$, since A_{N-1} is not Schur-Cohn, there exists a vector $\omega^* \in \mathbb{R}^n$ (an eigenvector of one of the non-converging natural modes) such that the solution to (11) from $\bar{x}^*(0) = \begin{bmatrix} 0^\top & \dots & 0^\top & \omega^{*\top} \end{bmatrix}^\top$ corresponds to $\bar{x}^*(t) = \begin{bmatrix} 0^\top & \dots & 0^\top & \bar{x}_N^\top(t) \end{bmatrix}^\top$, where $\bar{x}_N(t)$ does not converge to zero. As a consequence, the function in (15) along this solution corresponds to:

$$V(x^*(t)) = V((T\otimes I_n)\bar{x}^*(t)) = \bar{x}_N^\top(t)P_N\bar{x}_N(t),$$

which, from linearity, remains bounded away from zero. Then, using the first inequality in (18) we have that $|x^*(t)|_{\mathcal{A}}$ is bounded away from zero, namely solution $x^*(t)$ does not converge to the consensus set. In other words, the components of $x^*(t)$ do not asymptotically synchronize, which contradicts item (iv). ∎

III. Consensus in Quality-Fair video delivery

A. Background

In this paper we analyze the model considered in [4], in which delivery of video streams is performed to several mobile users sharing the same wireless resource. Quality fairness is is targeted, *i.e.*, the system aims at ensuring at each time instant the same utility value for all video streams.

The dynamics of the i-th video stream, $i = 1,\dots,N$, is described by the following set of equations (conveniently reported from [4, equation (22)]):

$$a_i(j)^+ = a_i(j) + \delta a_i(j) \tag{21a}$$
$$a_i^d(j)^+ = a_i(j) \tag{21b}$$
$$\Phi_i(j)^+ = \Phi_i(j) + \Delta U_i^{dd}(j) - U_i^{dd}(j) \tag{21c}$$
$$\Pi_i^b(j)^+ = \Pi_i^b(j) + (B_i(j) - B_0) \tag{21d}$$
$$R_i^{ed}(j)^+ = R_0 - \tfrac{K_P^{eb}+K_I^{eb}}{T}(B_i(j)-B_0) - \tfrac{K_I^{eb}}{T}\Pi_i^b(j) \tag{21e}$$
$$R_i^{edd}(j)^+ = R_i^{ed}(j) \tag{21f}$$
$$U_i^{dd}(j)^+ = f(a_i^d(j), R_i^{ed}(j)) \tag{21g}$$
$$B_i^+(j) = B_i(j) + [R_i^{edd}(j) - R_0 \tag{21h}$$
$$+ (K_P^t + K_I^t)\Delta U_i^{dd}(j) - K_I^t\Phi_i(j)]T$$
$$\bar{U}^{dd}(j) = \tfrac{1}{N}\sum_{k=1}^{N} U_k^{dd}(j) \tag{21i}$$

The discrete-time nonlinear state-space representation in (21) considers N mobile users, indexed by the subscript i, connected to the same base station (BS) and sharing wireless resources provided by the BS to get streamed videos delivered by N remote servers. Time is assumed to be slotted with a period T. Each video delivery chain is assumed to be controlled in a synchronous way, with video streams consisting of group of pictures (GoP) of the same duration T. Control is performed in a media-aware network element (MANE). The rate-utility function of the j-th GoP of the i-th stream is modeled by a nonlinear function $U_i(j) = f(a_i(j), R)$ of the video encoding rate R, parametrized by a vector $a_i(j)$, which value depends on the video characteristics. The evolution of $a_i(j)$ is described by (21a), with $\delta a_i(j)$ representing some uncontrolled perturbation modeling the variations with time of the rate-utility characteristics. A total transmission rate R_c is assumed to be shared by the users. The encoding rate target is evaluated within the MANE using an internal PI controller (controller K_{int}) aiming at regulating the buffer level B_i of the i-th stream around some reference buffer level B_0, see (21d) and (21e). K_P^{eb} and K_I^{eb} are the proportional and integral control parameters for the encoding rates. $R_0 = R_c/N$ is the average rate, which would be allocated in a rate-fair scenario. The draining rate of the i-th buffer within the MANE is controlled so as to minimize the discrepancy $\Delta U_i(j)$ of the utility $U_i(j)$ of the i-th program with respect to the average utility given by (21i). For that purpose, an external PI controller (controller K_{ext}) with parameters K_P^t and K_I^t is involved: programs with a utility less than average are drained faster, leading to an increase of the encoding rate. A one-period forward and backward delay between the MANE and the server is considered to account for moderate queuing delays in the network. Provided that T is of the order of the second, this is a realistic upper bound. The delay operators account for these delays in (21). In [3], [4] the four PI controller gains are heuristically selected to ensure the asymptotic convergence of the utilities $U_i(j)$ in (21g) to a common value $\bar{U}$, namely:

$$\lim_{j\to+\infty} U_i(j) = \bar{U}, \quad \forall i = 1,\dots,N. \tag{22}$$

Based on the results of Section II, we will provide below a systematic optimal selection of these parameters that ensures (22), as established in Theorem 2 in the forthcoming Section III-B.

B. Two PI control loops

System (21) can be rearranged in order to highlight the different contributions of two PI controllers. The first one essentially rejecting the constant bias B_0, and the second one rejecting the constant bias R_0 and inducing consensus of the utilities of the video streams. The first PI controller (denoted by K_{int} in Figure 1) corresponds to an internal loop and is characterized by (21d) and (21e), rewritten as:

$$\Pi_i^{b+} \quad = \quad \Pi_i^b + \Delta B_i \tag{23a}$$

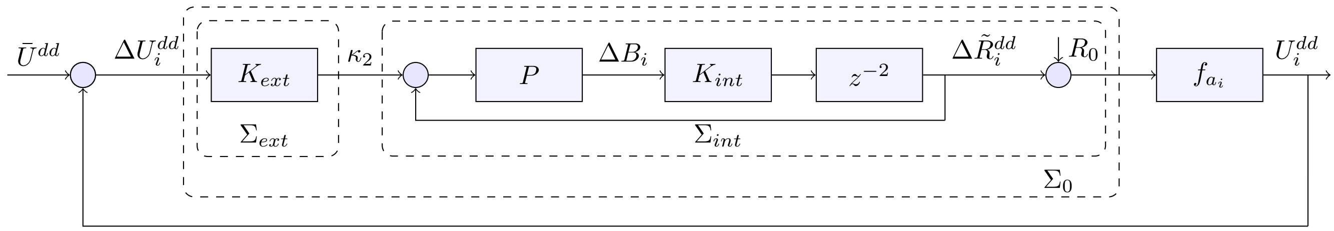


Fig. 1: Block Diagram of the controlled system

$$\kappa_1 \quad = \quad \tfrac{k_I^{int}}{T}\Pi_i^b + \tfrac{k_P^{int}}{T}\Delta B_i, \tag{23b}$$

where Π_i^b is the controller state, $\Delta B_i = B_i - B_0$ is the controller input and $\kappa_1 = -\Delta R_i^e = -(R_i^e - R_0)$ is the controller output. The integral and proportional gains k_I^{int} and k_P^{int} are defined as:

$$k_I^{int} = K_I^{eb}\,, \quad k_P^{int} = K_P^{eb} + K_I^{eb}. \tag{24}$$

The second PI controller (denoted by K_{ext} in Figure 1) is characterized by (21c) and (21h), rewritten as:

$$\Phi_i^{s+} \quad = \quad \Phi_i^s + \frac{\Delta U_i^{dd}}{\sigma} \tag{25a}$$

$$\kappa_2 \quad = \quad k_I^{ext}\Phi_i^s + \frac{k_P^{ext}}{\sigma}\Delta U_i^{dd} \tag{25b}$$

$$\Delta U_i^{dd} \quad = \quad \tfrac{1}{N}\textstyle\sum_{k=1}^N U_k^{dd} - U_i^{dd} \tag{25c}$$

where $\sigma > 0$ is a normalizing constant, $\Phi_i^s = \frac{\Phi_i}{\sigma}$ is the controller state vector, ΔU_i^{dd} is the controller input, and κ_2 is the controller output. The integral and proportional gains k_I^{ext} and k_P^{ext} are defined as:

$$k_I^{ext} = \sigma K_I^t\,, \quad k_P^{ext} = \sigma(K_P^t + K_I^t), \tag{26}$$

With this notation, (21e), (21f) and (21h) become:

$$\Delta R_i^{ed+} \quad = \quad \Delta R_i^e = -\kappa_1 \tag{27a}$$

$$\Delta R_i^{edd+} \quad = \quad \Delta R_i^{ed} \tag{27b}$$

$$\Delta B_i^+ \quad = \quad \Delta B_i + T(\Delta R_i^{edd} - \kappa_2). \tag{27c}$$

According to (23), (27) and as represented in Figure 1, controller K_{int} performs a delayed negative feedback action over the plant through the delayed output ΔR_i^{dd}.

C. The system seen as a consensus feedback

Let $\Sigma_{ext} = (A_{ext}, B_{ext}, C_{ext}, D_{ext})$ denote the state-space representation for controller K_{ext} (*i.e.*, the system with input variable ΔU_i^{dd} and output variable κ_2), and $\Sigma_{int} = (A_{int}, B_{int}, C_{int}, D_{int})$ denote the state-space representation for the feedback loop that includes the controller K_{int} (*i.e.*, the system with input variable κ_2 and output variable ΔR_i^{edd}). Then, using (23) and (27) for Σ_{int} and (25) for Σ_{ext}, one may represent the dynamics of the i-th video stream using the states x_{int} and x_{ext} defined as:

$$x_{int} = \begin{bmatrix} \frac{\Delta B_i}{T} & \frac{\Pi_i}{T} & -\Delta R_i^{ed} & -\Delta R_i^{edd} \end{bmatrix}^\top,\ x_{ext} = \Phi_i. \tag{28}$$

With this selection, the state-space matrices of the subsystems are given by:

$$\left(\begin{array}{c|c} A_{ext} & B_{ext} \\ \hline C_{ext} & D_{ext} \end{array}\right) = \left(\begin{array}{c|c} 1 & \frac{1}{\sigma} \\ \hline k_I^{ext} & \frac{k_P^{ext}}{\sigma} \end{array}\right), \tag{29}$$

$$\left(\begin{array}{c|c} A_{int} & B_{int} \\ \hline C_{int} & D_{int} \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ -k_P^{int} & -k_I^{int} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \end{array}\right). \tag{30}$$

According to Figure 1, one can then represent the inner dynamics of each video stream, represented by Σ_0 in Figure 1, as the cascaded interconnection of Σ_{ext} and Σ_{int}, establishing the linear relation from ΔU_i^{dd} to $\Delta R_i^{dd} + R_0 = R_i^{dd}$, whose state-space representation $\Sigma_0 = (A_0, B_0, C_0, D_0)$ is such that the state matrix A_0 is lower-triangular. Actually, given the state vector $x = \begin{bmatrix} x_{ext}^\top & x_{int}^\top \end{bmatrix}^\top$, the input variable ΔU_i^{dd} and the output variable ΔR_i^{dd} we have:

$$\left(\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right) = \left(\begin{array}{cc|c} A_{ext} & 0 & B_{ext} \\ B_{int}C_{ext} & A_{int} & B_{int}D_{ext} \\ \hline 0 & C_{int} & 0 \end{array}\right). \tag{31}$$

Due to its lower block triangular structure the eigenvalues of A_0 are the union of the eigenvalues of A_{int} and A_{ext}. Then the overall system dynamics is influenced by the separate actions of the two subsystems Σ_{int} and Σ_{ext}. In particular Σ_{int} performs an internal stabilizing action of each stream dynamics, and Σ_{ext} performs the external synchronization among the streams over the network.

D. Main consensus theorem

The coupling among the different video streams arises from the action of the average utility $\bar{U}^{dd}$ in (21i), acting as an input to each video stream dynamics, where the utility U_i^{dd} of each stream is a nonlinear function of the state a_i in (21a) and (21b). In particular, it is easily shown that (21g) leads to:

$$U_i^{dd} = f(a_i^{dd}, R_i^{edd}) = f(a_i^{dd}, \Delta R_i^{edd} + R_0), \tag{32}$$

so that U_i^{dd} can be seen as a nonlinear time-varying output of system Σ_0 in (31). In this paper we make the following strong assumption, so that a linear time-invariant analysis of the consensus algorithm can be performed.

Assumption 1: For each $i = 1, \ldots, N$, the input δa_i in (21a) is zero, so that a_i is constant for each i. Moreover there exist scalars h_i, $i = 1, \dots, N$ and a scalar $K_f > 0$ such that:

$$\begin{aligned} U_i^{dd} &= f(a_i^{dd}, R_i^{edd}) = h_i + K_f R_i^{edd} \\ &= h_i + K_f R_0 + K_f \Delta R_i^{edd}, \ \forall i = 1, \dots, N. \end{aligned} \quad (33)$$

Intuitively speaking, K_f translates the variation of utility provided by a variation of the video encoding rate. Based on Assumption 1 and on the presence of the integral action of controller K_{ext}, we may perform a coordinate change to compensate for the action of the constant disturbance $h_i + K_f R_0$, so that the overall system can be written as an output feedback network interconnection of N identical linear systems:

$$\begin{aligned} x_i^+ &= A_0 x_i + B_0 \Delta U_i^{dd} \\ U_i^{dd} &= K_f C_0 x_i \end{aligned} \quad \forall i = 1, \dots, N. \quad (34)$$

In particular, using (25c), each input ΔU_i^{dd} can be expressed, for each $i = 1, \dots, N$, as:

$$\Delta U_i^{dd} = \bar{U}^{dd} - U_i^{dd} = \frac{1}{N} \sum_{j \neq i} U_j^{dd} - \left(1 - \frac{1}{N}\right) U_i^{dd}. \quad (35)$$

Define now the vectors $U^{dd} = \begin{bmatrix} U_1^{dd} & \dots & U_N^{dd} \end{bmatrix}^\top$ and $\Delta U^{dd} = \begin{bmatrix} \Delta U_1^{dd} & \dots & \Delta U_N^{dd} \end{bmatrix}^\top$. Then (35), for $i = 1, \dots, N$, can be rewritten in the compact form:

$$\Delta U^{dd} = - \begin{bmatrix} 1-\frac{1}{N} & -\frac{1}{N} & \dots & \dots & -\frac{1}{N} \\ -\frac{1}{N} & 1-\frac{1}{N} & \dots & \dots & -\frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots & \\ -\frac{1}{N} & -\frac{1}{N} & \dots & & 1-\frac{1}{N} \end{bmatrix} U^{dd} = -L U^{dd}, \quad (36)$$

where L is the $N \times N$ *Laplacian matrix* associated with the network. The Laplacian matrix resumes the information exchanged by the subsystems. Notice that the graph related to the network described by matrix L defined in (36) is *fully connected*, *i.e.*, every vertex has an edge to every other vertex [11]. Combining (34) and (35), we obtain the following compact form for the overall system:

$$\begin{aligned} x^+ &= (I_N \otimes A_0) x + (I_N \otimes B_0)(-Ly) && (37a) \\ y &= U^{dd} = K_f (I_N \otimes C_0) x, && (37b) \end{aligned}$$

where y is the output representing the N utilities and $x = \begin{bmatrix} x_1^\top \dots x_N^\top \end{bmatrix}^\top$ is the overall state of the interconnected systems. Then the following theorem can be stated.

Theorem 2: Under Assumption 1, the following statements are equivalent:

(i) For any initial condition, all utilities U_i, $i = 1, \dots, N$ of model (23), (25), (27), (33) converge to the same value, *i.e.*, condition (22) is satisfied.

(ii) Given any solution to (37), there exists $\bar{U} \in \mathbb{R}$ such that $\lim_{j \to +\infty} y_i(j) = \bar{U}$, $\forall i = 1, \dots, N$.

(iii) The consensus set $\mathcal{A}$ in (3) is uniformly globally exponentially stable for dynamics (37) and matrix A_{int} is Schur-Cohn.

(iv) Matrix A_{int} and matrix

$$A_f = A_0 - K_f \left(\frac{N}{N-1} \right) B_0 C_0 \quad (38)$$

are both Schur-Cohn.

Proof: we will prove the theorem according to the following steps: (i) $\Leftrightarrow$ (ii), (iii) $\Leftrightarrow$ (iv), (iv) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iv).

Proof of (i) $\Leftrightarrow$ (ii). This equivalence follows from the fact that, due to relations (34)–(36), and from the definitions in (29)–(33), model (37) coincides with the closed loop (23), (25), (27), (33).

Proof of (iii) $\Leftrightarrow$ (iv). Applying the equivalence between items (i) and (iii) of Theorem 1 when focusing on system (37), item (iii) of Theorem 2 is equivalent to having that all eigenvalues λ_k of matrix L in (36), except for that one related to the eigenvector $\mathbf{1}_N$, are such that $A_0 - \lambda_k K_f B_0 C_0$ is Schur-Cohn. Since all the eigenvalues (except that one equals to zero) of the Laplacian matrix L are equal to $\frac{N}{N-1}$, the result trivially follows.

Proof of (iv) $\Rightarrow$ (i). Similar to the previous step, this implication follows from item (iv) of Theorem 1 after noticing that system $x_\circ^+ = A x_\circ$ corresponds to system Σ_0, namely $A = A_0$, where A_0 is given in (31). Since A_{int} is Schur-Cohn by assumption, then due to its block triangular structure, matrix A_0 has a single eigenvalue at zero and all solutions to (7) converge to a constant, thereby proving item (i) of Theorem 2.

Proof of (i) $\Rightarrow$ (iv). We prove this by contradiction. Assume that item (iv) does not hold. Then either A_f is not Schur-Cohn, which implies from Theorem 1 that consensus is not achieved for some initial conditions (thereby proving that (i) does not hold), or A_f is Schur-Cohn and A_{int} is not Schur-Cohn. In this case, Theorem 1 applies because A_f is Schur-Cohn and all solutions exponentially synchronize to a solution to (7) with $A = A_0$ as in (31). Then two cases may occur:

a) A_{int} has at least one eigenvalue with magnitude larger than 1 or at least one eigenvalue on the unit circle with multiplicity larger than 1: in this case some solutions synchronize to a diverging evolution, thus item (i) does not hold;

b) A_{int} has at least one eigenvalue with magnitude 1 on the unit disk. If that eigenvalue is at 1, then due to the triangular structure, matrix A_0 has two eigenvalues in 1 (the other one coming from A_{ext}) and again some solutions synchronize to a diverging evolution. If that eigenvalue is anywhere else in the unit circle, then it generates a revolving non-constant mode and some solutions synchronize to a non-convergent oscillatory evolution. In both cases **a)** and **b)**, item (i) does not hold and the proof is completed. ■

IV. Control Design

In this section we address the problem of finding suitable gains K_{ext}, K_{int} in order to guarantee i) the asymptotic stability of matrices A_f and A_{int} in item (iv) of Theorem 2 and ii) the optimization of the system performance in terms of the convergence rate of A_{int} and A_{ext}. The basic idea consists in designing the PI controller through a numerical technique based on an iterative LMI approach. The proposed algorithm allows addressing general static output feedback design problems, and can be viewed as an alternative approach to coordinate descent algorithms [1], [9]. The gain selection consists in a two-steps optimization process in which first the controller K_{int} is designed in order to maximize the convergence rate of A_{int}, and once K_{int} is fixed, the same procedure will be applied to the selection of K_{ext} in order to maximize the convergence rate of A_f. Let us first note that, after a suitable permutation of the state variables, matrices A_{int} in (29) and A_f in (38) can be conveniently rewritten as a function of the controllers $K_{int} = \begin{bmatrix} k_P^{int} & k_I^{int} \end{bmatrix}$ and $K_{ext} = \begin{bmatrix} k_I^{ext} & k_P^{ext} \end{bmatrix}$ as follows:

$$A_{int} = A_1 - B_1 K_{int} \left[I \quad 0 \right], \tag{39}$$

$$A_f = A_2 - B_2 K_{ext} \left[I \quad 0 \right], \tag{40}$$

where we have defined:

$$\left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right) = \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array} \right), \tag{41}$$

$$\left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right) = \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -k_I^{int} & 0 & -k_P^{int} & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right). \tag{42}$$

A possible solution to this problem is given in Algorithm 1. The core idea is to alternate between two main steps, each of them requiring the solution of a quasi-convex optimization problem, *i.e.*, a generalized eigenvalue problem (GEVP) based on LMI and bisection, where the controller gain K and a multiplier G are alternated as optimization variables. Algorithm 1 joins several useful properties that make it a promising tool for computing suboptimal selections of the static output feedback gain. Clearly, there is no guarantee of optimality, as the static output feedback problem is well known to be a challenging and nonconvex one. Some useful properties of Algorithm 1 are stated and proven next.

Proposition 1: The following statements hold:

(i) *Initialization and termination*: Given any input $(A, B, [I \quad 0])$ and $\delta > 0$, the Initialization step in Algorithm 1 gives an admissible pair, and the algorithm terminates after a finite number of steps.

(ii) *Feasibility*: Given any admissible pair $(\alpha_{L_1}, \alpha_{H_1})$ from Step 1, the pair $(\alpha_{L_2}, \alpha_{H_2})$ obtained from the subsequent Step 2 always satisfies $\alpha_{L_2} \geq \alpha_{L_1}$, and viceversa.

(iii) *Guarantees*: Any solution (K_{out}, α_{out}) resulting from Algorithm 1 satisfies $\sigma\left(A - BK_{out}\left[I \quad 0\right]\right) \subseteq \mathbb{C}_{\leq \beta}$, where $\beta = \sqrt{1 - \alpha_{out}}$. In particular, if $\alpha_{out} > 0$, then the gain selection K_{out} is a stabilizing output feedback gain for the triple $(A, B, [I \quad 0])$, and β is the corresponding convergence rate.

Proof: Proof of (i). First, we prove that $(\alpha_0, 1.1)$ is an admissible pair in the sense clarified in the initialization step. Trivially, (44) is infeasible with $\alpha > 1$, because the upper-left entry is positive. To show that (44) is feasible with $\alpha = \alpha_L = \sigma_0$ as in (43), select $G_{11} = I$, $G_{22} = I$, $G_{21} = 0$, $X_1 = 0$, and $W = I$ so that, applying a Schur complement, (44) is feasible if:

$$\left(\sigma_0 - 1\right) I + AA^\top \leq 0, \tag{47}$$

Algorithm 1 Convergence rate α and controller K

Input: Matrices A, B, $C = [I \quad 0]$, and a tolerance $\delta > 0$.
Initialization: Set $M = 0$ and initialize the pair $(\alpha_L, \alpha_U) = (\sigma_0, 1.1)$, where, using $\bar{\sigma}(A)$ to denote the maximum singular value of A, we select:

$$\sigma_0 = 1 - \bar{\sigma}^2(A). \tag{43}$$

Pair (α_L, α_U) is *admissible* for (44), in the sense that (44) is feasible with $\alpha = \alpha_L$ and infeasible with $\alpha = \alpha_U$.
Iteration

Step 1: Given M and pair (α_L, α_H) from the previous step, solve, using bisection with tolerance $\delta > 0$, the GEVP:

$$\begin{aligned} &\max_{W, G_{11}, G_{21}, G_{22}, X_1, \alpha} \quad \alpha \\ &\text{s.t.} \quad \begin{bmatrix} -W + \alpha W & AG - BX \\ \star & -G - G^\top + W \end{bmatrix} \leq 0, \end{aligned} \tag{44}$$

where $W = W^\top > 0$ and matrices, G and X have the following structure (see, for example, [26] for details on the use of multipliers):

$$G = \begin{bmatrix} G_{11} & G_{11}M \\ G_{21} & G_{22} \end{bmatrix}, X = X_1 \left[I \quad M \right]. \tag{45}$$

In particular, determine an admissible pair (α_L, α_H) such that $\alpha_H - \alpha_L \leq \delta$. Pick the (sub)optimal solution $\bar{G}_{11}$, $\bar{X}_1$ corresponding to α_L, and set $\bar{K} = \bar{G}_{11}^{-1} \bar{X}_1$ for the next step.

Step 2: Given $\bar{K}$ and pair (α_L, α_U) from the previous step, set $\bar{A} = A - B\bar{K}C$, and solve, using bisection with tolerance $\delta > 0$, the GEVP:

$$\begin{aligned} &\max_{\alpha, W = W^\top > 0} \quad \alpha \\ &\text{s.t.} \quad \bar{A} W \bar{A}^\top - W \leq -\alpha W. \end{aligned} \tag{46}$$

In particular, determine an admissible pair (α_L, α_H) such that $\alpha_H - \alpha_L \leq \delta$. Pick the (sub)optimal solution $\bar{W} = \begin{bmatrix} \bar{W}_{11} & \bar{W}_{12} \\ \bar{W}_{21} & \bar{W}_{22} \end{bmatrix}$ (where W has the partition induced by G), corresponding to α_L and set $M = \bar{W}_{11}^{-1} \bar{W}_{12}$ for the next step.
until α_L does not increase more than δ over three consecutive steps.
Output: $K_{out} = \bar{K}$ and $\alpha_{out} = \alpha_L$.

---

which is clearly ensured if $\sigma_0 - 1 + \bar{\sigma}^2(A) \leq 0$. We now prove that the algorithm always terminates in a finite number of steps. Let $\alpha_{L_1}^j$ denote the value of α_L at the j-th iteration of Step 1. From item (ii) of Proposition 1 the sequence $\alpha_{L_1}^j$, $j \in \mathbb{N}$, is non decreasing and upper bounded by $\alpha = 1$, thus it is convergent, *i.e.*, given $\delta > 0$ there exists an index $j \in \mathbb{N}$ such that $\alpha_{L_1}^{j+1} - \alpha_{L_1}^j \leq \delta$.

Proof of (ii). [From Step 1 to Step 2]. By substituting the solution α_{L_1}, $\bar{K}$ obtained from Step 1 in (44) we get that:

$$\begin{bmatrix} -W + \alpha_{L_1} W & (A - B\bar{K}C)G \\ \star & -G - G^\top + W \end{bmatrix} \leq 0 \tag{48}$$

has a feasible solution. By applying Finsler's Lemma, feasibility of (48) is equivalent to feasibility of:

$$(A - B\bar{K}C)W(A - B\bar{K}C)^\top - W \leq -\alpha_{L_1} W. \tag{49}$$

Comparing (49) with (46), it follows that the subsequent solution α_{L_2} to Step 2 satisfies $\alpha_{L_2} \geq \alpha_{L_1}$.
[From Step 2 to Step 1]. Substitute the solution α_{L_2}, M obtained from Step 2 in (46) and perform a Schur complement to get:

$$\begin{bmatrix} -W + \alpha_{L_2} W & (A - BKC)W \\ \star & -W - W^\top + W \end{bmatrix} \leq 0, \tag{50}$$

which corresponds to (44) with $W = G$. It follows that the subsequent solution α_{L_1} to Step 1 satisfies $\alpha_{L_1} \geq \alpha_{L_2}$.

Proof of (iii). From linear systems theory [14], we get that both solutions at Step 1 and Step 2 provide a certificate that matrix $\bar{A} = A - BK_{out}C$ has a spectral radius smaller than α_{out}. ■

Remark 1: There is no loss of generality in considering systems is the form $(A, B, [I \quad 0])$ in Algorithm 1. For a system in a general form (A, B, C), where matrix C is full-row rank, there always exists a nonsingular matrix T such that $CT^{-1} = [I \quad 0]$. Using T as a similarity transformation we obtain $\big(T^{-1}AT, T{-}1B, CT\big) = \big(\bar{A}, \bar{B}, [I \quad 0]\big)$.

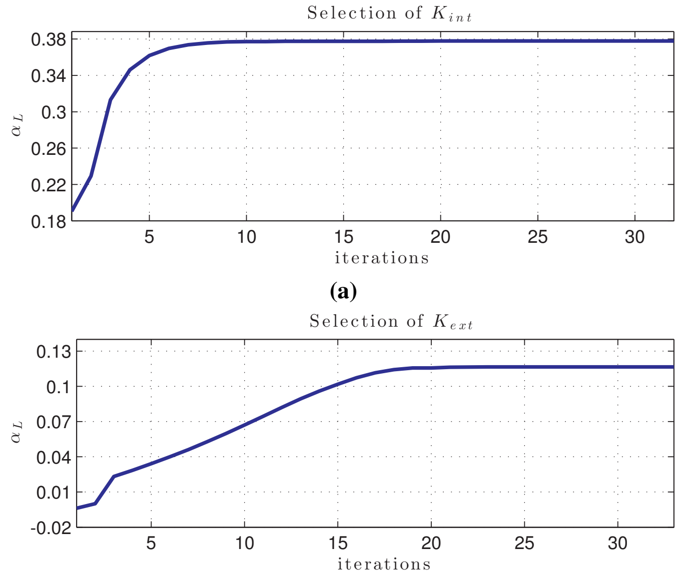
To demonstrate the effectiveness and the convergence of the proposed algorithm, the outer and inner loop gains K_{int} and K_{ext} of the application discussed on Section III are designed using the general procedure in Algorithm 1, with tolerance $\delta = 10^{-8}$, and with the selections in (41) and (42), respectively. Figure 2a shows that after 32 iterations, the rate α_L related to the selection of K_{int} (see (39), (41)) corresponds to 0.37789, with $K_{int} = [0.19256 \quad 0.012915]$. Figure 2b shows similar results for the selection of K_{ext}: after 33 iterations the value of α_L is 0.1165 with $K_{ext} = [0.17645 \quad 0.65801]$. In Table I the results of the algorithm in comparison with [8] are summarized. In our preliminary work [8] we used a graphical method to solve the disign problem for K_{int} and K_{ext} with the same performance goal as that one of Algorithm 1. Table I shows that Algorithm 1 provides similar results to those of [8] by using a systematic approach, which can be generalized to systems of any order, contrarily to the graphical method in [8].

Method	k_P^{int}	k_I^{int}	α_L
[8]	0.2	0.0145	0.365747
Algorithm 1	0.19256	0.012915	0.37789
Method	k_P^{ext}	k_I^{ext}	α_L
[8]	0.6590	0.1765	0.1166
Algorithm 1	0.65801	0.17645	0.1165

TABLE I: Comparision between the PI controller gains K_{int} and K_{ext} provided in [8] and those obtained from Algorithm 1.

V. SIMULATION RESULTS

In this section we provide simulation results for the control scheme (21) characterized in Theorem 2, with the gain selection in (24), (26) and Table I. This selection guarantees item (iv) (therefore all other items) of Theorem 2, as



(b)

Fig. 2: Gain Selection via Algorithm 1. Evolution of α_L through the iterations: **(a)** A_{int} and **(b)** A_f.

established in Proposition 1 of Section IV. To verify the performance of the control design in Table I, six video streams[1] of different types have been encoded during 60 s with x.264 [47] in 4CIF (704×576) format at various bit rates. The programs are Interview (Prog 1), Sport (Prog 2), Big Buck Bunny (Prog 3), Nature Documentary (Prog 4), Video Clip (Prog 5), and an extract of *Spiderman* (Prog 6). The frame rate is $F = 30$ frames/s. GoPs of 10 frames are considered, thus the GoP duration is $T = 0.33$ s. The considered utility U_i is the Peak Signal-to-Noise Ratio (PSNR). To tune the controllers, the rate-utility characteristics of each GoP is estimated as described in [3], [4]. The control is assumed to be performed within the MANE, closely located to the BS to which the clients are connected. A Matlab simulation of the behavior of the servers, the network, the MANE, and the clients is performed. The forward and backward propagation and queuing delays between the MANE and the servers are taken as constant and equal to T. The packets delivered by the MANE to the BS and to the clients are assumed to be well received thanks to retransmission at the MAC layer, which is not modeled here. During the control of the streaming system, the rate-utility characteristics are *not* available at the MANE. Only the utility of the encoded packets it receives are used. They may be tagged, *e.g.*, at the RTP layer of the protocol stack. The MANE adjusts the transmission rate of each stream and provides an encoding rate target to the individual servers, which are then responsible of meeting this target by video encoding, transcoding, or bit-rate switching. Model (21) developed in [3] has been simulated setting the parameters as follows. We have chosen $B_0 = 1200$ kb to

[1] http://www.youtube.com/watch?v=l2Y5nIbvHLs, =G63TOHluqno, =YE7VzlLtp-4, =NNGDj9IeAuI, =rYEDA3JcQqw, =SYFFVxcRDbQ.

tolerate significant variations of the buffering delay, and the channel rate is $R_0 = 4000$ kb/s.

Two triplets of simulations are presented in Figures 3 and 4, corresponding to different selection of $N = 4$ video streams out of the six ones described above. In particular, each figure shows comparatively the evolution of the utilities U_i, namely, as commented before, the PSNR of the streams, for three selections of the control parameters K_I^{eb}, K_P^{eb}, K_I^t, and K_P^t in (21), reported in Table II and arising from:

(a) Subfigures **(a)** (left). *Tuning based on Theorem 2 and Algorithm 1.* The value of the parameter K_f introduced in (33) has to be selected. To this end, the time and ensemble average of the rate-PSNR characteristics for the four first streams have been evaluated at different constant encoding rates ranging from $R^e = 250$kb/s to $R^e = 2$Mb/s. The resulting values of K_f range from $k_f = 0.02$ dB/kb/s to $k_f = 0.0025$ dB/kb/s. To avoid too aggressive variations of the video encoding rate and increase robustness of the system, $K_f = 0.02$ dB/kb/s has been selected. Considering $N = 4$, the PI gains in (21) can be chosen using the values in Table I within (24) and (26), with $\sigma = K_f N/(N-1) = 0.0267$.

(b) Subfigures **(b)** (middle). *Tuning based on [3].* The parameters proposed in [3] reported in Table II that correspond to the selection $K_{int} = [0.152 \quad 0.002]$, $K_{ext} = [0.418 \quad 0.5944]$. With this selection one can easily verify that matrix A_{int} is Schur-Cohn, but A_f has unstable eigenvalues. Then Theorem 2 anticipates lack of consensus. This is confirmed by the middle plots of Figures 3 and 4[2].

(c) Subfigures **(c)** (right). *Tuning based on a Transmission Rate-Fair (TRF) scheme.* In the TRF scheme, no external controller is applied, so that the transmission rate is always the same for all programs, while the internal controller gains are set as in [3]. Also in this case Theorem 2 anticipates lack of consensus, because matrix A_f has poles at the limit of stability (indeed the scheme corresponds to $L = \mathbf{0}_{N,N}$, with only zero eigenvalues). A straightforward extension of Theorem 1 predicts lack of convergence but also lack of convergence of the utilities.

The first four video streams, which have been used for the tuning of K_f, are considered in the experiments reported in Figure 3, showing the utilities of the four users. Figure 4 shows the utilities in another simulation using the last four videos. With the aforementioned schemes, the average absolute value of the difference of the PSNR of each stream and the average PSNR is evaluated as follows:

$$\overline{\Delta U} = \frac{1}{MN} \sum_{j=1}^{M} \sum_{k=1}^{N} \left| U_k\left(j\right) - \overline{U}\left(j\right) \right|. \tag{51}$$

Table III captures the results of the simulations presented in Figures 3 and 4. We notice that the presented scheme gives better results in terms of average utility in comparison with the other control strategies. In particular, since the scheme in [3] does not stabilize the consensus set, it is not surprising that the utilities obtained with this scheme diverge when sending the last four videos (see Figure 4b), and we were nopt able to evaluate the corresponding average utility.

[2]Note that the gains in [3] had been heuristically tuned without any formal guarantee of convergence.

Method	K_I^{eb}	K_P^{eb}	K_I^t	K_P^t
Algorithm 1	0.012922	0.1797	$0.418/\sigma$	$0.17645/\sigma$
Scheme [3]	0.002	0.15	0.05	100
TRF Scheme	0.002	0.15	0	0

TABLE II: Comparison between the gain tuning for the control schemes.

	Algorithm 1	TRF scheme	Scheme [3]
Progs 1−4	2.66	4.48	2.96
Progs 3−6	3.55	3.98	−

TABLE III: Comparision of the average utility $\bar{\Delta U}$ obtained with different control schemes (the values are in dB).

VI. Concluding Remarks

Consensus for multi-agent systems has been characterized in the case of N identical (continuous- or discrete-time) systems connected through a network of fixed topology. Necessary and sufficient conditions have been proven to ensure uniform global exponential stability of the state consensus set. The results have been applied in the context of the quality-fair delivery of videos and general LMI-based iterative procedure addressing design of static output feedback linear stabilizers has been proposed to design (sub)-optimally the PI controller gains. Future research directions include the extension of the methodology to obtain alternative performance guarantees. Furthermore, it might be useful to adapt the results to the case where the input of the system is limited in magnitude (saturation).

Appendix A
Proof of Lemma 1

Before proving Lemma 1, let us introduce some useful results. The following Lemma is based on [39, Theorem 1.10].

Lemma 2: Given a closed, convex set $\mathcal{A} \subset \mathbb{R}^\nu$ and any vector $x \in \mathbb{R}^\nu$, there exists a unique point $y \in \mathcal{A}$ satisfying:

$$|x - y| = |x|_{\mathcal{A}} := \min_{a \in \mathcal{A}} |x - a|. \tag{52}$$

Moreover, $y \in \mathcal{A}$ satisfies (52) if and only if $x \in N_{\mathcal{A}}(y)$, where:

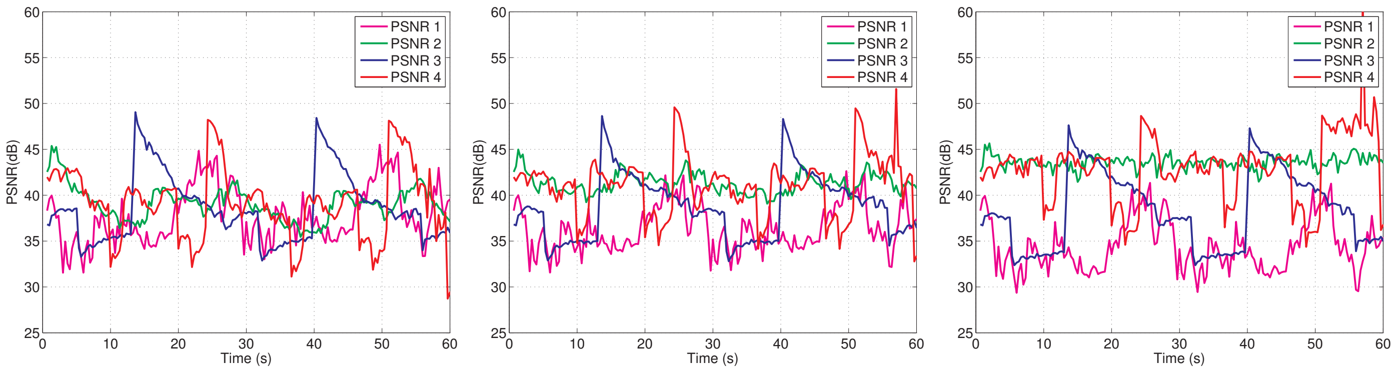
$$N_{\mathcal{A}}(y) = \{n \in \mathbb{R}^\nu : \langle n - y, y - a\rangle \geq 0 \quad \forall a \in \mathcal{A}\} \tag{53}$$

is the normal cone to $\mathcal{A}$ at y, and y is the orthogonal projection of x onto $\mathcal{A}$ (see [30]).

Proof: we only prove the equivalence among (52) and (53) because the existence and uniqueness of y is already proven in [31, Theorem 12.3].

Proof of (53) $\Rightarrow$ (52). If $x \in N_{\mathcal{A}}(y)$ then, $\forall a \in \mathcal{A}$ we have:

$$\begin{aligned} |x-a|^2 &= |x-y+y-a|^2 \\ &= |x-y|^2 + |y-a|^2 + 2\langle x-y, y-a\rangle \\ &\geq |x-y|^2. \end{aligned}$$



(a) Control parameters proposed here (Table I). **(b)** Control parameters taken from [3]. **(c)** Transmission rate fair streaming.

Fig. 3: PSNR of Progs 1 to 4, comparison between different control schemes.

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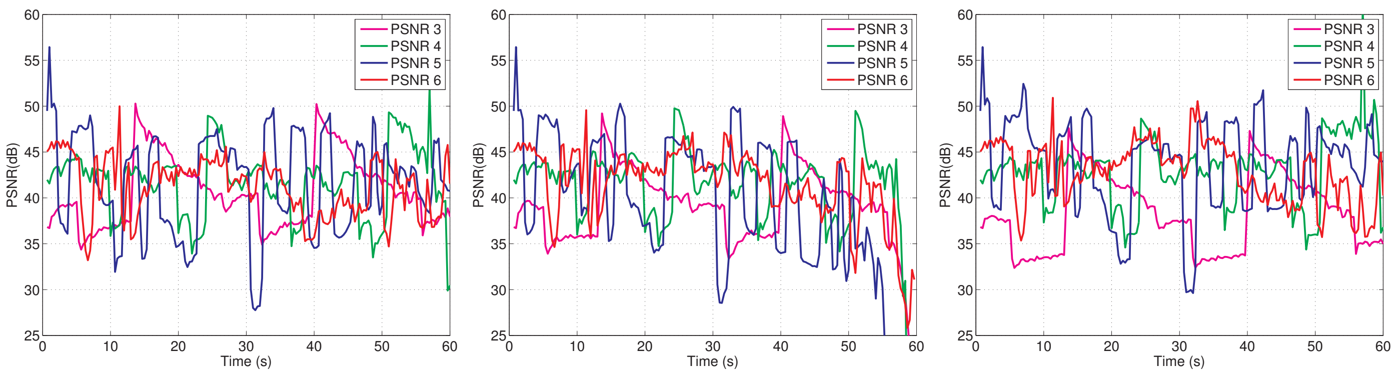

(a) Control parameters proposed here (Table I). **(b)** Control parameters taken from [3]. **(c)** Transmission rate fair streaming.

Fig. 4: PSNR of Progs 3 to 6, comparison between different control schemes.

Proof of (52)$\Rightarrow$ (53). For all $a \in \mathcal{A}$ and for any $\eta \in (0,1]$ we have from convexity that $\eta a + (1-\eta)y \in \mathcal{A}$, therefore from (52):

$$\begin{aligned} |x-y|^2 &\le |x-(\eta a+(1-\eta)y)|^2 \\ &= |x-y-\eta(a-y)|^2 \\ &= |x-y|^2 + 2\eta\langle x-y, y-a\rangle + \eta^2 |y-a|^2, \end{aligned}$$

which, dividing by η, implies:

$$2\langle x-y, y-a\rangle + \eta |y-a|^2 \ge 0.$$

Taking the limit as $\eta \to 0$, the statement is proven. ■

Using Lemma 2 we can prove the following:

Lemma 3: For any pair of positive integers n, N, given set $\mathcal{A}$ in (3), we have for all $x \in \mathbb{R}^{Nn}$:

$$|x|^2_{\mathcal{A}} = \sum_{k=1}^{N} |\bar{x} - x_k|^2, \text{ with } \bar{x} := \frac{1}{N}\sum_{k=1}^{N} x_k = \frac{1}{N}(\mathbf{1}_N^\top \otimes I_n)x, \tag{54}$$

where $x_k \in \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$ is the (vector) average of the (vector) components of $x \in \mathbb{R}^{Nn}$.

Proof: let us select $y = \mathbf{1}_N \otimes \bar{x} \in \mathbb{R}^{nN}$, so that $|x-y|^2 = \sum_{k=1}^{N} |x_k - \bar{x}|^2$. Then, according to Lemma 2, the proof is completed if $x \in N_{\mathcal{A}}(y)$. To prove this fact, first note that, since $\mathcal{A}$ is a linear subspace, for any pair of vectors $y, a \in \mathcal{A}$, we have $b := y - a \in \mathcal{A}$, so that it is enough to show:

$$\langle x-y, b\rangle \ge 0, \quad \forall b \in \mathcal{A}. \tag{55}$$

Relation (55) can be established by first noticing that $b \in \mathcal{A}$ implies that there exists $\bar{b} \in \mathbb{R}^n$ such that $b = \mathbf{1}_N \otimes \bar{b}$, and then computing:

$$\begin{aligned} \langle x-y, b\rangle &= \langle x - \mathbf{1}_N \otimes \bar{x}, \mathbf{1}_N \otimes \bar{b}\rangle \\ &= \langle \mathbf{1}_N \otimes \bar{b}, x - \mathbf{1}_N \otimes \bar{x}\rangle \\ &= (\mathbf{1}_N \otimes \bar{b})^\top \left(x - \mathbf{1}_N \otimes \frac{1}{N}(\mathbf{1}_N^\top \otimes I_n)x \right) \\ &= \frac{1}{N}(\mathbf{1}_N^\top \otimes \bar{b}^\top)\left(N I_{Nn} - \mathbf{1}_N \otimes \mathbf{1}_N^\top \otimes I_n\right)x \\ &= \frac{1}{N}(\mathbf{1}_N^\top \otimes \bar{b}^\top)\left(\left[N I_N - \mathbf{1}_N \otimes \mathbf{1}_N^\top\right] \otimes I_n\right)x \end{aligned}$$

$$= \frac{1}{N}\left(\underbrace{\mathbf{1}_N^\top\left[NI_N - \mathbf{1}_N \otimes \mathbf{1}_N^\top\right]}_{=0} \otimes \bar{b}^\top\right) x = 0$$

which completes the proof. ■

Proof of Lemma 1: based on Lemma 3, we can now prove Lemma 1. Since matrix Δ has a zero in the upper left entry and ones in the remaining diagonal entries, we can write:

$$T\Delta T^\top = \bar{T}\bar{T}^\top \tag{56}$$

where $\bar{T} \in \mathbb{R}^{N\times(N-1)}$, composed by the last $N-1$ columns of T, satisfies $\bar{T}^\top \mathbf{1}_N = 0$ and has $N-1$ independent columns. Therefore, Im $\bar{T} \subset (\mathbf{1}_N)^\perp$. As a consequence Im $\begin{bmatrix} 1 & 1 & \dots & 1 \\ -1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{bmatrix} \subset$ Im $\bar{T}$, and there exists Σ invertible such that $(\Sigma\bar{T}^\top \otimes I_n)x = \tilde{x} := \begin{bmatrix} x_1 - x_2 \\ \vdots \\ x_1 - x_N \end{bmatrix} \in \mathbb{R}^{(N-1)n}$, where $\tilde{x}$ clearly satisfies $\sum_{k=2}^{N}(x_1 - x_k)^2 = |\tilde{x}|^2$. From relation (56) consider now the quadratic form:

$$x^\top(T\Delta T^\top \otimes I_n)x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}^\top (\bar{T}\bar{T}^\top \otimes I_n) \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

$$= \left((\Sigma\bar{T}^\top \otimes I_n)\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}\right)^\top (M \otimes I_n) \left((\Sigma\bar{T}^\top \otimes I_n)\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} I_n & -I_n & 0 & \dots & 0 \\ I_n & 0 & -I_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ I_n & 0 & 0 & \dots & -I_n \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}\right)^\top$$

$$(M \otimes I_n)\left(\begin{bmatrix} I_n & -I_n & 0 & \dots & 0 \\ I_n & 0 & -I_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ I_n & 0 & 0 & \dots & -I_n \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}\right) = \tilde{x}^\top(M \otimes I_n)\tilde{x},$$

where $\tilde{x} = \begin{bmatrix} x_1^\top - x_2^\top & \dots & x_1^\top - x_N^\top \end{bmatrix}^\top$. Then noticing that $\lambda_{min}(M \otimes I_n) = \lambda_{min}(M) = c_1$ and $\lambda_{max}(M \otimes I_n) = \lambda_{max}(M) = c_2$ we obtain the inner inequalities in (3). To complete the proof we need to show the outer inequalities in (3). To this end, it is sufficient to show that there exist positive scalars k_1 and k_2 such that for any pair n, N and any $x \in \mathbb{R}^{Nn}$:

$$k_1 |\tilde{x}|^2 \le \sum_{k=1}^{N} |\bar{x} - x_k|^2 \le k_2 |\tilde{x}|^2, \tag{57}$$

and then the result follows from Lemma 3. To show (57) we first observe that $\sum_{k=1}^{N} |\bar{x} - x_k|^2 = |\bar{x} \otimes \mathbf{1}_n - x|^2$ and then the straightforward relation:

$$\tilde{x} = \underbrace{\begin{bmatrix} I_n & -I_n & 0 & \dots & 0 \\ I_n & 0 & -I_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ I_n & 0 & 0 & \dots & -I_n \end{bmatrix}}_{T_1}(\bar{x} \otimes \mathbf{1}_n - x) \tag{58}$$

implies $|\tilde{x}|^2 = \tilde{x}^\top\tilde{x} = (\bar{x} \otimes \mathbf{1}_n - x)^\top T_1^\top T_1(\bar{x} \otimes \mathbf{1}_n - x) \le k_1^{-1} |\bar{x} \otimes \mathbf{1}_n - x|$, where k_1^{-1} is the maximum singular value of $T_1^\top T_1$. Similarly we have:

$$\begin{aligned} &\frac{1}{N}\left[-I_n \quad -I_n \quad \dots \quad -I_n\right]\tilde{x} = \\ &= \frac{1}{N}\left(-(N-1)x_1 + \sum_{k=2}^{N} x_k + x_1 - x_1\right) \\ &= \frac{1}{N}\left(-Nx_1 + \sum_{k=1}^{N} x_k\right) = \bar{x} - x_1 \end{aligned} \tag{59}$$

which implies:

$$\begin{aligned} &\frac{1}{N}\left[(N-1)I_n \quad -I_n \quad \dots \quad -I_n\right]\tilde{x} = \\ &= \bar{x} - x_1 + \frac{N}{N}(x_1 - x_2) = \bar{x} - x_2 \end{aligned} \tag{60}$$

Using similar reasonings, one gets:

$$(\bar{x} \otimes \mathbf{1}_n - x) = \underbrace{\begin{bmatrix} -I_n & -I_n & -I_n & \dots & -I_n \\ (N-1)I_n & -I_n & -I_n & \dots & -I_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -I_n & -I_n & -I_n & \dots & (N-1)I_n \end{bmatrix}}_{T_2}\tilde{x}, \tag{61}$$

which implies $|\bar{x} \otimes \mathbf{1}_n - x|^2 = (\bar{x} \otimes \mathbf{1}_n - x)^\top(\bar{x} \otimes \mathbf{1}_n - x) = \tilde{x}^\top T_2^\top T_2\tilde{x} \le k_2 |\tilde{x}|^2$, where k_2 is the maximum singular value of $T_2^\top T_2$. ■

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